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A COMPARATIVE STUDY OF SHELL EQUATIONS BY THREE-DIMENSIONAL FEM ANALYSIS

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Abstract—An analysis of the accuracy of thin shell equations is made for both classical and refined equations. The comparison is based on an exact reference obtained by a three-dimensional procedure using a finite element method (FEM). The classical equations are here represented by those of Flügge and Morley–Koiter. A higher approximation of the constitutive shell equations is used in the refined equations. The investigation is done by means of natural frequencies for free undamped vibrations of axisymmetric shells, especially circular cylindrical shells. In order to investigate the *a priori* estimate of the intrinsic errors of the thin shell equations an error expansion is employed, i.e. the error is written in powers of the thickness to principal curvature ratio h/R and thickness to characteristic wavelength ratio h/L. The results hereby obtained indicate that the error estimate agrees with the true error. Furthermore, the classical equations yield remarkably good results compared with the refined equations. The natural frequencies of shells in the shape of a spherical zone with two boundaries obtained by classical thin shell equations and FEM are briefly compared. These results also indicate agreement between the error estimate and the true error.

1. INTRODUCTION

Reducing the number of dimensions in structural analysis is an often used method to obtain a solvable problem; examples are beam, plate and shell theory. Such theories require approximations and hence we must accept errors in the predicted results compared with an exact analysis.

It is well known, Koiter[1] and Niordson[2], that a linear thin shell theory based on Love's first approximation of the strain energy as the sum of a quadratic term in the membrane strain measure and a quadratic term in the bending strain measure of the shell has inherent errors of, at most, the relative order $(h/R) + (h/L)^2$ where h is the thickness of the shell, R is the numerical value of the smallest principal radius of curvature, and L is a characteristic wavelength of the deformation pattern of the middle surface. If we add to the strain energy the mixed term of the membrane and the bending strain measure we get, as pointed out in Ref. [2], a relative error of, at most, the order $(h/R)^2 + (h/L)^2$. The penalty for this improved accuracy is more complicated relations between the two strain measures on the one hand and the membrane stress and the moment on the other. They are now no longer uncoupled as they were in the case of Love's first approximation for the strain energy. For the general case, the coupled stress–strain relations or constitutive equations are presented in Niordson[3]. We use these to establish refined constitutive equations for the case of a circular cylindrical shell.

Our goal is to investigate the accuracy of thin shell equations, i.e. the *a priori* estimates of the errors. The shell equations to be used are the classic equations of Flügge and Morley–Koiter and the refined equations in Ref. [3]. The analysis is done by means of an error expansion in powers of h/R and h/L, i.e.

$$\frac{\omega_s - \omega_e}{\omega_s} = A\frac{h}{R} + B\left(\frac{h}{R}\right)^2 + C\frac{h}{L} + D\left(\frac{h}{L}\right)^2 + E\left(\frac{h^2}{RL}\right) \tag{1}$$

where ω_s is the natural frequency predicted by thin shell theory and ω_e is the true or exact frequency of the structure. We shall argue that we are able to obtain ω_e by a three-dimensional finite element method (FEM), here applied to axisymmetric shells. The coefficients A–E are evaluated in the case of free undamped vibrations of a circular cylinder with free boundaries. We have chosen the natural frequencies of free vibrations as an

evaluation measure because here we get a global scalar response of the structure.

Our results confirm that the *a priori* error estimates agree well with the true error, at least for the shells under consideration. The results obtained from the classical equations are also found to be remarkably good compared with the results from the more complicated refined equations.

2. SHELL ASSUMPTIONS AND INTRINSIC ERRORS

We define a shell as a three-dimensional body bounded by two arbitrary but sufficiently smooth outer surfaces a relatively small distance apart compared with other characteristic dimensions of the body. The behaviour of the body is completely described by the middle surface which is the locus of points equidistant from the boundary surfaces.

Throughout the context we shall apply tensor notation. Here, we use the notation that lower case Latin indices have the range 1–3, while Greek letter indices have the range 1, 2. The summation convention for repeated indices are also used.

Establishing an elastic thin shell theory requires a number of assumptions on the behaviour of the structure. These are often "based" on the Love–Kirchhoff assumptions which, following the formulations of Ref. [3], state

(1) that points which lie on one and the same normal to the undeformed middle surface also lie on one and the same normal to the deformed middle surface;

(2) that points which lie on one and the same surface parallel to the undeformed middle surface lie on one and the same surface parallel to the deformed middle surface.

With these assumptions we are able to provide a two-dimensional representation of the original three-dimensional problem.

The reduction procedure can be separated into three parts. First, we link displacements and strains. Secondly, we set up equations of equilibrium using the principle of virtual work. Finally, we need constitutive equations that link together strains and stresses. It is possible to go through the first two steps without further assumptions than those of Love–Kirchhoff, i.e. these equations are fully exact in that sense. But introducing the constitutive equations makes it necessary to specify the shell material and to include some further approximations.

The derivation of the linear part of the connection between the middle surface tangential and normal displacements v^α and w and the strain measures denoted the strain tensor $E_{\alpha\beta}$ and the bending tensor $K_{\alpha\beta}$ can be found in Koiter[4] and Niordson[5] and are

$$E_{\alpha\beta} = \frac{1}{2}(D_\alpha v_\beta + D_\beta v_\alpha) - d_{\alpha\beta} w \tag{2}$$

$$K_{\alpha\beta} = D_\alpha D_\beta w + d_{\alpha\gamma} D_\beta v^\gamma + d_{\beta\gamma} D_\alpha v^\gamma + v^\gamma D_\beta d_{\gamma\alpha} - d_{\beta\gamma} d^\gamma_\alpha w \tag{3}$$

where D_α denotes covariant differentiation and $d_{\alpha\beta}$ is the second fundamental tensor of the middle surface.

The equilibrium equations are

$$\begin{aligned} D_\alpha N^{\alpha\beta} + 2d^\beta_\gamma D_\alpha M^{\gamma\alpha} + M^{\gamma\alpha} D_\alpha d^\beta_\gamma + F^\beta = 0 \\ D_\alpha D_\beta M^{\alpha\beta} - d_{\alpha\gamma} d^\gamma_\beta M^{\alpha\beta} - d_{\alpha\beta} N^{\alpha\beta} - p = 0 \end{aligned} \tag{4}$$

where the symmetric tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are the effective membrane stresses and moments, respectively. F^β is the external load vector in the tangential plane of the middle surface and p is the external normal load.

We now focus on the derivation of the constitutive equations. They are obtained from the strain energy W per unit area of the middle surface. To do this we need to accept that:

(a) the material is linear elastic, homogeneous and isotropic;
(b) the shell is thin, i.e. $h/R \ll 1$;
(c) the strains are small everywhere;
(d) the state of stress is approximately plane;
(e) the thickness h is constant.

From the above assumptions we have a material that obeys Hooke's law. Therefore, the strain energy must be a quadratic function of the strain measures, i.e.

$$\frac{W}{Eh} = C^{\alpha\beta\gamma\delta}E_{\alpha\beta}E_{\gamma\delta} + D^{\alpha\beta\gamma\delta}E_{\alpha\beta}K_{\gamma\delta} + F^{\alpha\beta\gamma\delta}K_{\alpha\beta}K_{\gamma\delta} + r \tag{5}$$

where E is Young's modulus and the fourth-order tensors $C^{\alpha\beta\gamma\delta}$, $D^{\alpha\beta\gamma\delta}$ and $F^{\alpha\beta\gamma\delta}$ are functions of the geometry of the shell and Poisson's ratio ν. The last term r is a quadratic term in the derivatives of $E_{\alpha\beta}$ and $K_{\alpha\beta}$ of all orders. In order to simplify eqn (5) Ref. [2] examines the individual terms and the errors introduced by the simplification.

The result of the examination is as follows. If we omit terms of order $(h/R)^2$ in the first and third terms of eqn (5) it can be written as

$$C^{\alpha\beta\gamma\delta} = C_{10}a^{\alpha\beta}a^{\gamma\delta} + C_{20}a^{\alpha\gamma}a^{\beta\delta} \tag{6}$$

$$F^{\alpha\beta\gamma\delta} = F_{10}a^{\alpha\beta}a^{\gamma\delta} + F_{20}a^{\alpha\gamma}a^{\beta\delta} \tag{7}$$

where $a^{\alpha\beta}$ is the first fundamental tensor of the middle surface. If we omit terms of like order in the mixed term of eqn (5) it can be written as

$$\begin{aligned} D^{\alpha\beta\gamma\delta} = {} & D_{10}Hh^2a^{\alpha\beta}a^{\gamma\delta} + D_{20}Hh^2a^{\alpha\gamma}a^{\beta\delta} + D_{30}h^2a^{\alpha\beta}d^{\gamma\delta} \\ & + D_{40}h^2a^{\gamma\delta}d^{\alpha\beta} + D_{50}h^2a^{\alpha\gamma}d^{\beta\delta} \end{aligned} \tag{8}$$

where H is the mean curvature of the middle surface. If we then omit $D^{\alpha\beta\gamma\delta}$ totally we introduce a relative error of, at most, the order h/R. Love's first approximation for the strain energy does this. Therefore, if we retain these terms of eqn (5), the error will be of higher order.

Neglecting the last term r in eqn (5) introduces an error of, at most, the relative order $(h/L)^2$, where L is a characteristic wavelength in the deformation pattern of the middle surface.

Applying the principle of virtual work we obtain the constitutive equations by mere derivation, i.e.

$$N^{\alpha\beta} = \frac{\partial W}{\partial E_{\alpha\beta}}; \qquad M^{\alpha\beta} = \frac{\partial W}{\partial K_{\alpha\beta}}. \tag{9}$$

With the assumptions, eqns (6)–(8), we obtain after some rather laborious algebra[3]

$$\begin{aligned} N^{\alpha\beta} = {} & \frac{Eh}{1+\nu}(E^{\alpha\beta} + \tilde{\nu}a^{\alpha\beta}E^{\gamma}_{\gamma})\left(1 + \frac{Kh^2}{12}\right) \\ & + \frac{Eh^3}{24(1+\nu)}(-3(d^{\alpha\gamma}K^{\beta}_{\gamma} + d^{\beta\gamma}K^{\alpha}_{\gamma}) + 2d^{\delta}_{\delta}K^{\alpha\beta} \\ & - 4\tilde{\nu}a^{\alpha\beta}d^{\gamma\delta}K_{\gamma\delta} - 5\tilde{\nu}d^{\alpha\beta}K^{\gamma}_{\gamma} + \tilde{\nu}(2 - 3\tilde{\nu})d^{\delta}_{\delta}a^{\alpha\beta}K^{\gamma}_{\gamma}) \end{aligned} \tag{10}$$

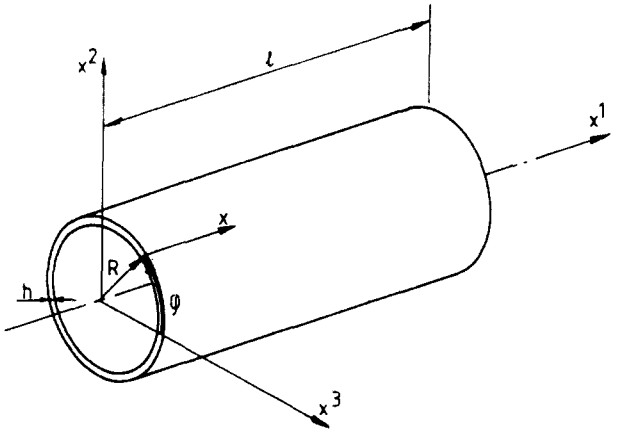

Fig. 1. Sketch of the cylinder.

and

$$M^{\alpha\beta} = \frac{Eh^3}{24(1+\nu)}(2K^{\alpha\beta} + 2\tilde{\nu}a^{\alpha\beta}K^{\gamma}_{\gamma} - 3(d^{\alpha\gamma}E^{\beta}_{\gamma} + d^{\beta\gamma}E^{\alpha}_{\gamma}) + 2d^{\delta}_{\delta}E^{\alpha\beta} - 4\tilde{\nu}d^{\alpha\beta}E^{\gamma}_{\gamma} - 5\tilde{\nu}a^{\alpha\beta}d^{\gamma\delta}E_{\gamma\delta} + \tilde{\nu}(2 - 3\tilde{\nu})d^{\delta}_{\delta}a^{\alpha\beta}E^{\gamma}_{\gamma}) \tag{11}$$

where K is the Gaussian curvature of the middle surface and

$$\tilde{\nu} = \frac{\nu}{1-\nu}. \tag{12}$$

With this we *a priori* introduce an error of the relative order $(h/R)^2 + (h/L)^2$.

3. VIBRATION OF A THIN CYLINDRICAL SHELL

We consider a uniform thin circular cylindrical shell of length l, mean radius R and thickness h, depicted in Fig. 1. A point on the middle surface is defined by the axial coordinate x ranging from 0 to l and the arc length ϕ.

The fundamental tensors are

$$a^{\alpha\beta} = a_{\alpha\beta} = \delta^{\alpha}_{\beta} \tag{13}$$

$$d_{11} = d_{12} = d_{21} = 0, \qquad d_{22} = -1/R \tag{14}$$

where δ^{α}_{β} is Kronecker's delta. The displacement field is u, v and w in the axial, tangential and radial (positive outwards) directions, respectively. From eqns (2) and (3) we obtain the strain measures and from eqns (10) and (11) we derive the constitutive equations which are presented in the Appendix. With these we get the equations of equilibrium from eqn (4). In the case of free vibration the external loads are given by the d'Alembert forces. Here, it is important that we retain terms of the order $(h/R)^2$ in the derivation of the d'Alembert

forces because terms of this order are included in the equations of equilibrium. Doing so, we obtain

$$\left.\begin{aligned} F^1 &= -\rho h \frac{\partial^2}{\partial t^2}(u - kRw_{,x}) \\ F^2 &= -\rho h \frac{\partial^2}{\partial t^2}((1+6k)v - 3kRw_{,\phi}) \\ p &= -\rho h \frac{\partial^2}{\partial t^2}(w + kRu_{,x} + 3kRv_{,\phi} - kR^2 w_{,xx} - kR^2 w_{,\phi\phi}) \end{aligned}\right\} \tag{15}$$

where we have used a comma for partial differentiation, ρ is the density of the shell material and

$$k = \frac{h^2}{12R^2}. \tag{16}$$

For a complete cylinder, the general solution for free vibrations can be written as a Fourier expansion in the tangential direction and harmonics in time t, i.e.

$$\left.\begin{aligned} u &= \sum_{m=0}^{\infty} \sum_{j=1}^{8} u_{mj} \cos(m\phi/R) \exp(\alpha_{mj} x/R) e^{i\omega t} \\ v &= \sum_{m=0}^{\infty} \sum_{j=1}^{8} v_{mj} \sin(m\phi/R) \exp(\alpha_{mj} x/R) e^{i\omega t} \\ w &= \sum_{m=0}^{\infty} \sum_{j=1}^{8} w_{mj} \cos(m\phi/R) \exp(\alpha_{mj} x/R) e^{i\omega t} \end{aligned}\right\} \tag{17}$$

where m is the number of circumferential waves. The summation over j is due to the fulfilment of the boundary conditions. If we specify non-homogeneous boundary conditions, the natural frequencies ω will depend upon all the harmonics m, Forsberg[6]. Therefore, we shall only use homogeneous boundary conditions, in which case the summation over m in eqn (17) can be discarded. Substitution of eqns (17) and (15) into the equations of equilibrium (see Appendix) leads to a fourth-order algebraic equation in α_{mj}^2

$$A_1 \alpha_{mj}^8 + A_2 \alpha_{mj}^6 + A_3 \alpha_{mj}^4 + A_4 \alpha_{mj}^2 + A_5 = 0 \tag{18}$$

with

$$A_i = A_i(h/R, \nu, m, \Omega) \tag{19}$$

where

$$\Omega^2 = \frac{R^2(1-\nu^2)\rho}{E}\omega^2. \tag{20}$$

With this we can express u_{mj} and v_{mj} as functions of w_{mj}, i.e. only eight unknowns are involved. They are determined by specifying four homogeneous boundary conditions at each end of the cylinder. At our disposal are the membrane normal force N, the membrane

shear force S, the shear force Q, and the bending moment M or the kinematic conditions, i.e. we can prescribe

either N or u and
either S or v and
either Q or w and
either M or ψ

where $\psi = w_{,x}$ is the rotation of the boundary edge. The static conditions are derived from the principle of virtual work. In our case they are

$$\left.\begin{aligned} N &= N^{11} \\ S &= N^{12} - 2/RM^{12} \\ Q &= -M^{11}_{,x} - 2M^{12}_{,\phi} \\ M &= M^{11}. \end{aligned}\right\} \tag{21}$$

This leads to eight homogeneous linear equations in the unknown w_{mj}. The condition for a non-trivial solution is that the determinant vanishes. This gives us a transcendental equation from which the natural frequencies can be obtained.

4. FINITE ELEMENT MODELLING

The problem of structural eigenfrequencies in the FEM formulation is

$$[S][\Psi] = [M][\Psi][\Lambda] \tag{22}$$

where $[\Lambda]$ and $[\Psi]$ correspond to quadratic eigenfrequencies and eigenmodes, and $[S]$, $[M]$ are the global stiffness and mass matrix, respectively. We treat this problem with the subspace iteration, Bathe[7], a procedure to obtain the lowest p eigenfrequencies of a generalized eigenvalue problem, eqn (22), with a q-dimensional subspace, $p \leqslant q$.

In order to achieve reliable results from the FEM analysis we use a fully three-dimensional element. We shall not use a shell element because such an element includes the errors of the shell theory on which it is founded. We use an axisymmetric solid triangular element with a quadratic displacement assumption and a Fourier expansion in the tangential direction.

We have chosen an analytical derivation of the stiffness and mass matrices instead of just numerical integration because this procedure is faster and more stable in the numerical sense. The derivation is related to that of Pedersen and Cederkvist[8]. However, the element is not hampered by their assumption of an element side parallel to the symmetry axis.

Further details concerning the derivation of the stiffness and mass matrices can be found in an internal report, Ladefoged[9].

5. NUMERICAL RESULTS FOR A CYLINDRICAL SHELL

To check the validity of the method of solution as well as the algebra and programming, comparisons with results obtained by other investigators are presented. The frequency is given in the form

Table 1. The frequency coefficient c obtained by the refined equations (first row) and Ref. [10] (second row): (a) with both ends free and $m = 6$; (b) with both ends clamped and $m = 4$

(a)

$h/R = 0.01$				$h/R = 0.05$			
l/R				l/R			
3.15	3.19	4.90		1.60	3.41	4.71	5.23
0.283	0.122	0.283		0.703	0.704	0.518	0.703
0.283	0.123	0.283		0.707	0.707	0.520	0.707

(b)

$h/R = 0.01$				$h/R = 0.05$			
l/R				l/R			
1.03	1.98	2.78	3.35	1.14	2.43	2.92	3.24
0.405	0.197	0.412	0.100	0.539	0.282	0.546	0.245
0.412	0.200	0.412	0.100	0.548	0.283	0.548	0.245

Table 2. The frequency coefficient c obtained by the refined equations (first row) and Ref. [11] (second row). One end is clamped and the other is free and $h/R = 0.01$

	l/R					l/R			
m	1.18	2.16	3.00	5.29	m	1.39	3.49	4.34	5.93
2	0.282	0.123	0.283	0.283	4	0.100	0.100	0.0436	0.100
	0.283	0.123	0.283	0.283		0.100	0.100	0.0436	0.100
	l/R					l/R			
m	1.22	3.17	4.05	5.31	m	1.01	2.13	2.83	4.71
6	0.118	0.118	0.100	0.118	8	0.192	0.182	0.192	0.192
	0.118	0.118	0.100	0.118		0.193	0.182	0.193	0.193

$$\omega = \frac{c}{R}\sqrt{\left(\frac{E}{\rho(1-\nu^2)}\right)}. \tag{23}$$

Here, c is a dimensionless coefficient which is tabulated. We shall use Poisson's ratio $\nu = 0.3$ throughout. In Warburton[10] a similar approach is used, based on the equations of Flügge. The boundaries are free and clamped, respectively. In Table 1 the first row is from the refined equations and the second from Ref. [10]. In Sharma[11], a Rayleigh–Ritz procedure and Budiansky–Sanders "best" first-order theory are used to examine a clamped–free cylinder. In Table 2 the first row is from the refined equations and the second from Ref. [11].

Fairly good agreement is observed. It should be noted that the refined equations yield a lower frequency than the corresponding classical theory, although by a very small amount.

In order to justify that we are able to obtain natural frequencies close to the exact frequencies with FEM, we give in Fig. 2 the four lowest frequencies for a number of subdivisions of a cylinder.

We observe that the frequencies show a monotonically decreasing rate with increasing subdivision. Already subdivision No. 2 is fairly close to the "exact" solution from subdivision No. 5. On account of this we have used subdivisions similar to No. 3.

Now we return to our major goal. We present a study of the frequency coefficient c due to various values of the thickness to radius ratio h/R with fixed length to radius ratio l/R and number of circumferential waves m. The boundaries are free, i.e. all the boundary conditions are static. The results obtained from the refined equations are presented in Tables 3 and 4 for the modes which are symmetric and skew–symmetric with respect to the central cross-section of the cylinder, respectively, along with the results from classical equations. We have here used Flügge from Ref. [10] and Morley–Koiter from Koiter[12]. As pointed out by Koiter[13] the boundary condition (13) $N_{x\phi} = 0$ of a free edge in Ref. [10] is in error and should be replaced by $N_{x\phi} - 1/RM_{x\phi} = 0$ following the nomenclature

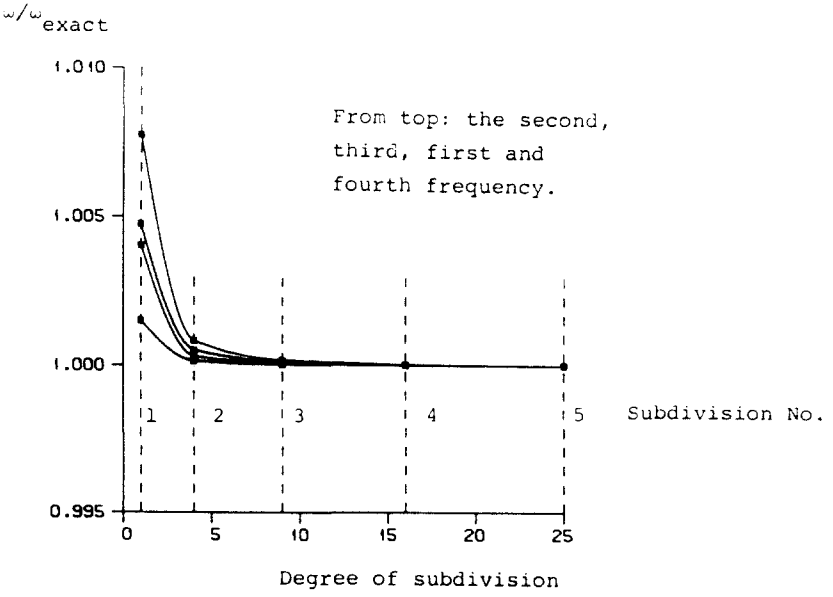

Fig. 2. FEM frequencies as functions of the subdivision.

Table 3. The frequency coefficient c (symmetric mode) obtained from the refined equations (first row), Flügge (second row), Morley–Koiter (third row) and FEM (fourth row), $l/R = 1.0$

	Number of circumferential waves, m								
h/R	0	1	2	3	4	5	6	7	8
0.02	0.9497	0.8782	0.0153	0.0434	0.0832	0.1345	0.1973	0.2715	0.3572
	0.9498	0.8784	0.0153	0.0434	0.0832	0.1345	0.1974	0.2717	0.3575
	0.9497	0.8783	0.0153	0.0434	0.0832	0.1345	0.1973	0.2716	0.3575
	0.9497	0.8783	0.0153	0.0433	0.0831	0.1343	0.1969	0.2708	0.3560
0.03	0.9497	0.8787	0.0230	0.0649	0.1245	0.2013	0.2953	0.4064	0.5348
	0.9499	0.8790	0.0230	0.0649	0.1245	0.2014	0.2956	0.4070	0.5359
	0.9498	0.8789	0.0230	0.0649	0.1245	0.2014	0.2955	0.4070	0.5358
	0.9499	0.8787	0.0230	0.0649	0.1242	0.2006	0.2938	0.4038	0.5303
0.04	0.9497	0.8792	0.0306	0.0864	0.1657	0.2679	0.3931	0.5411	0.7118
	0.9500	0.8796	0.0306	0.0864	0.1658	0.2682	0.3937	0.5424	0.7143
	0.9499	0.8795	0.0306	0.0864	0.1657	0.2681	0.3936	0.5423	0.7141
	0.9500	0.8791	0.0305	0.0862	0.1650	0.2663	0.3897	0.5348	0.7014
0.05	0.9496	0.8797	0.0381	0.1079	0.2068	0.3344	0.4906	0.6752	0.8879
	0.9500	0.8803	0.0381	0.1079	0.2069	0.3349	0.4919	0.6778	0.8926
	0.9499	0.8802	0.0381	0.1079	0.2069	0.3348	0.4917	0.6776	0.8925
	0.9501	0.8795	0.0381	0.1075	0.2055	0.3313	0.4841	0.6634	0.8683
0.08	0.9491	0.8815	0.0608	0.1719	0.3297	0.5330	0.7811	1.0733	1.4085
	0.9502	0.8828	0.0607	0.1720	0.3303	0.5351	0.7863	1.0839	1.4277
	0.9500	0.8828	0.0607	0.1719	0.3301	0.5348	0.7860	1.0835	1.4273
	0.9505	0.8806	0.0605	0.1704	0.3247	0.5208	0.7565	1.0291	1.3360
0.10	0.9486	0.8832	0.0758	0.2145	0.4112	0.6644	0.9724	1.3340	1.7472
	0.9504	0.8850	0.0758	0.2147	0.4125	0.6685	0.9826	1.3545	1.7843
	0.9500	0.8850	0.0757	0.2145	0.4122	0.6681	0.9821	1.3540	1.7837
	0.9494	0.8816	0.0753	0.2116	0.4018	0.6416	0.9269	1.2534	1.6167

of Flügge[14]. This is done here. The FEM solution is presented as well.

From Tables 3 and 4 we observe that all the shell equations over-estimate the frequency for all the parameters considered. As the number of circumferential waves m and thickness to radius ratio h/R increase the frequency coefficient obtained from Flügge

Table 4. The frequency coefficient c (skew–symmetric mode) obtained from the refined equations (first row), Flügge (second row), Morley–Koiter (third row) and FEM (fourth row), $l/R = 1.0$

	Number of circumferential waves, m								
h/R	0	1	2	3	4	5	6	7	8
0.02	0.9531	0.9379	0.0279	0.0694	0.1171	0.1729	0.2385	0.3145	0.4014
	0.9532	0.9383	0.0279	0.0694	0.1171	0.1730	0.2386	0.3147	0.4017
	0.9532	0.9380	0.0279	0.0694	0.1171	0.1730	0.2386	0.3147	0.4017
	0.9531	0.9375	0.0277	0.0691	0.1165	0.1721	0.2373	0.3129	0.3991
0.03	0.9531	0.9498	0.0417	0.1040	0.1755	0.2592	0.3573	0.4712	0.6012
	0.9532	0.9508	0.0417	0.1040	0.1755	0.2593	0.3577	0.4718	0.6024
	0.9532	0.9501	0.0417	0.1040	0.1755	0.2593	0.3577	0.4718	0.6024
	0.9529	0.9488	0.0414	0.1031	0.1740	0.2569	0.3539	0.4661	0.5939
0.04	0.9530	0.9662	0.0556	0.1385	0.2338	0.3452	0.4759	0.6273	0.8002
	0.9533	0.9679	0.0556	0.1386	0.2339	0.3456	0.4767	0.6289	0.8030
	0.9532	0.9668	0.0556	0.1385	0.2338	0.3455	0.4767	0.6289	0.8029
	0.9526	0.9643	0.0549	0.1369	0.2308	0.3406	0.4688	0.6166	0.7843
0.05	0.9529	0.9869	0.0694	0.1730	0.2919	0.4310	0.5941	0.7828	0.9980
	0.9533	0.9895	0.0694	0.1730	0.2921	0.4317	0.5957	0.7859	1.0035
	0.9532	0.9878	0.0693	0.1730	0.2921	0.4317	0.5956	0.7859	1.0034
	0.9522	0.9835	0.0683	0.1702	0.2869	0.4229	0.5813	0.7634	0.9692
0.08	0.9523	1.0713	0.1102	0.2754	0.4651	0.6867	0.9454	1.2437	1.5825
	0.9535	1.0780	0.1102	0.2755	0.4661	0.6895	0.9518	1.2562	1.6043
	0.9532	1.0740	0.1101	0.2753	0.4659	0.6892	0.9516	1.2560	1.6040
	0.9508	1.0598	0.1072	0.2672	0.4495	0.6598	0.9019	1.1763	1.4821
0.10	0.9518	1.1432	0.1369	0.3428	0.5794	0.8550	1.1759	1.5448	1.9620
	0.9536	1.1536	0.1369	0.3429	0.5811	0.8604	1.1884	1.5689	2.0039
	0.9532	1.1477	0.1367	0.3426	0.5807	0.8599	1.1878	1.5682	2.0031
	0.9508	1.1221	0.1321	0.3289	0.5522	0.8077	1.0988	1.4255	1.7855

and Morley–Koiter, respectively, coincide. The results from the refined equations are smaller. This is shown in Figs 3–5 where the percentage error defined as

$$\varepsilon = \frac{\omega_{\text{shell}} - \omega_{\text{FEM}}}{\omega_{\text{shell}}} 100\% \tag{24}$$

is depicted. Indeed, this behaviour is expected. In the case of the skew–symmetric mode, see Figs 3(b)–5(b), the trends from Figs 3(a)–5(a) are almost identical but amplified.

In the case of a circular cylinder, the smallest characteristic wavelength L of the middle surface becomes

$$L = \min\left(\frac{l}{n}, \frac{\pi R}{m}\right) \tag{25}$$

where n is the number of axial half-waves. However, we only use

$$L = \frac{\pi R}{m} \tag{26}$$

in the power expansion (1) because, for a fixed number of axial half-waves, the axial wavelength is kept constant and with that the influence. Coefficients A–E in eqn (1) are evaluated from Tables 3 and 4 by means of a least squares procedure. In this procedure we omit the cases $m = 0$ and 1. This is done because the corresponding mode shapes are fundamentally different from those of the higher wave numbers. For the three sets of shell equations under consideration we arrived at Table 5.

From this we observe that although the figures are different in the case of the symmetric and skew–symmetric mode, respectively, the trends are similar. The deviation can be explained by the fact that the axial influence is different in the two cases considered.

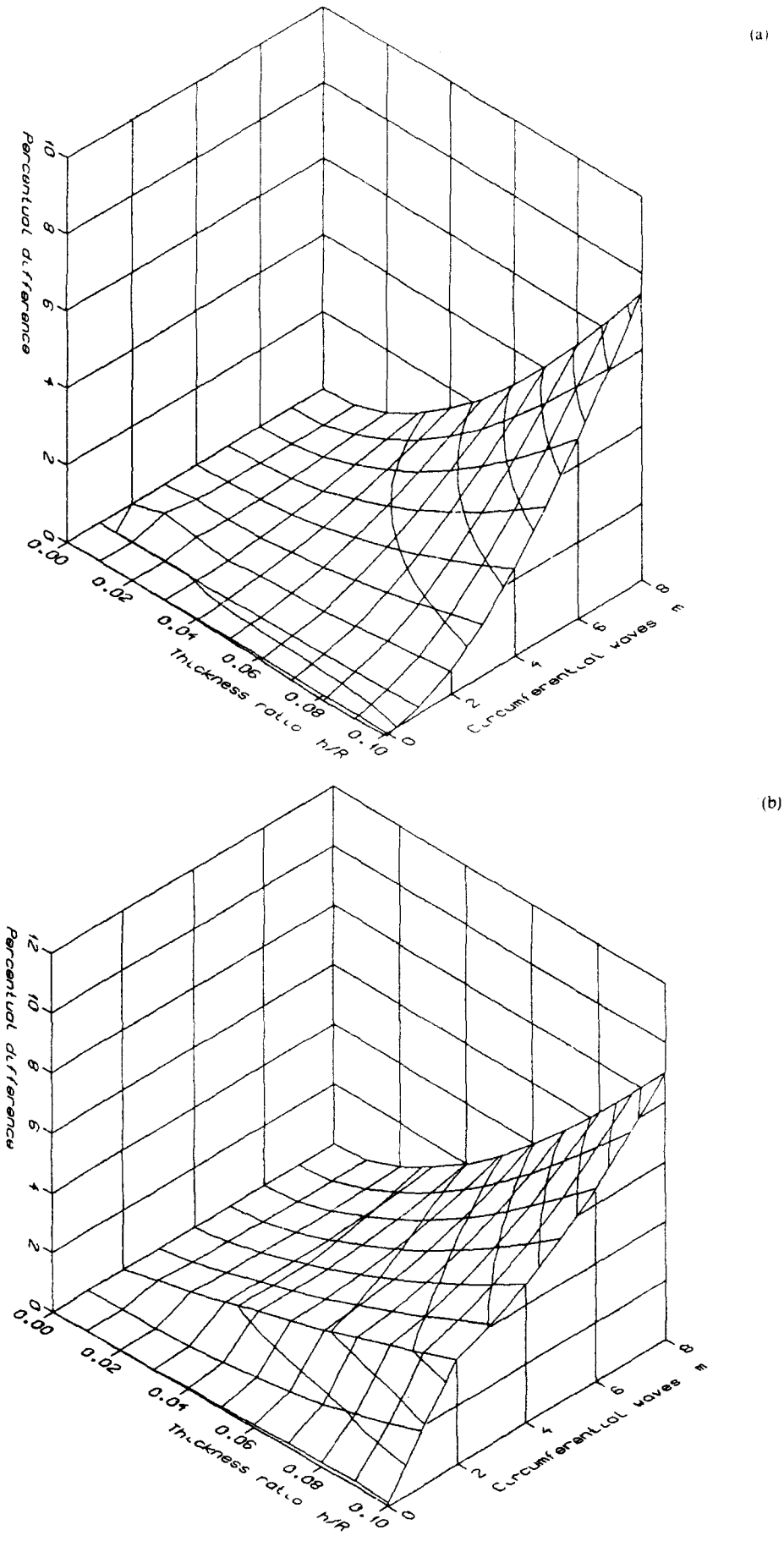

Fig. 3. Percentage errors ε obtained from the refined equations: (a) symmetric mode; (b) skew-symmetric mode. Percentage isolines with 1% spacing are shown.

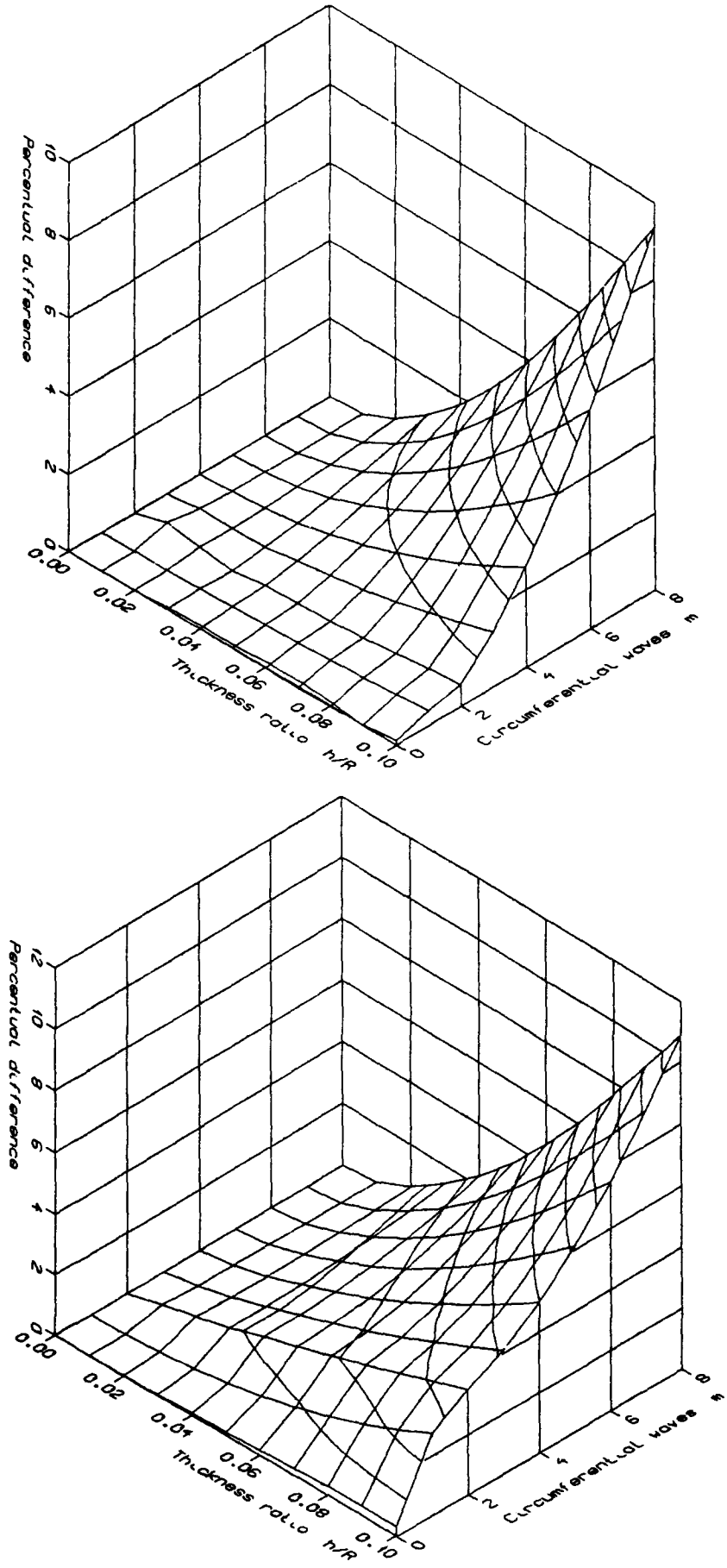

Fig. 4. Percentage errors ε obtained from the equations of Flügge: (a) symmetric mode; (b) skew–symmetric mode. Percentage isolines with 1% spacing are shown.

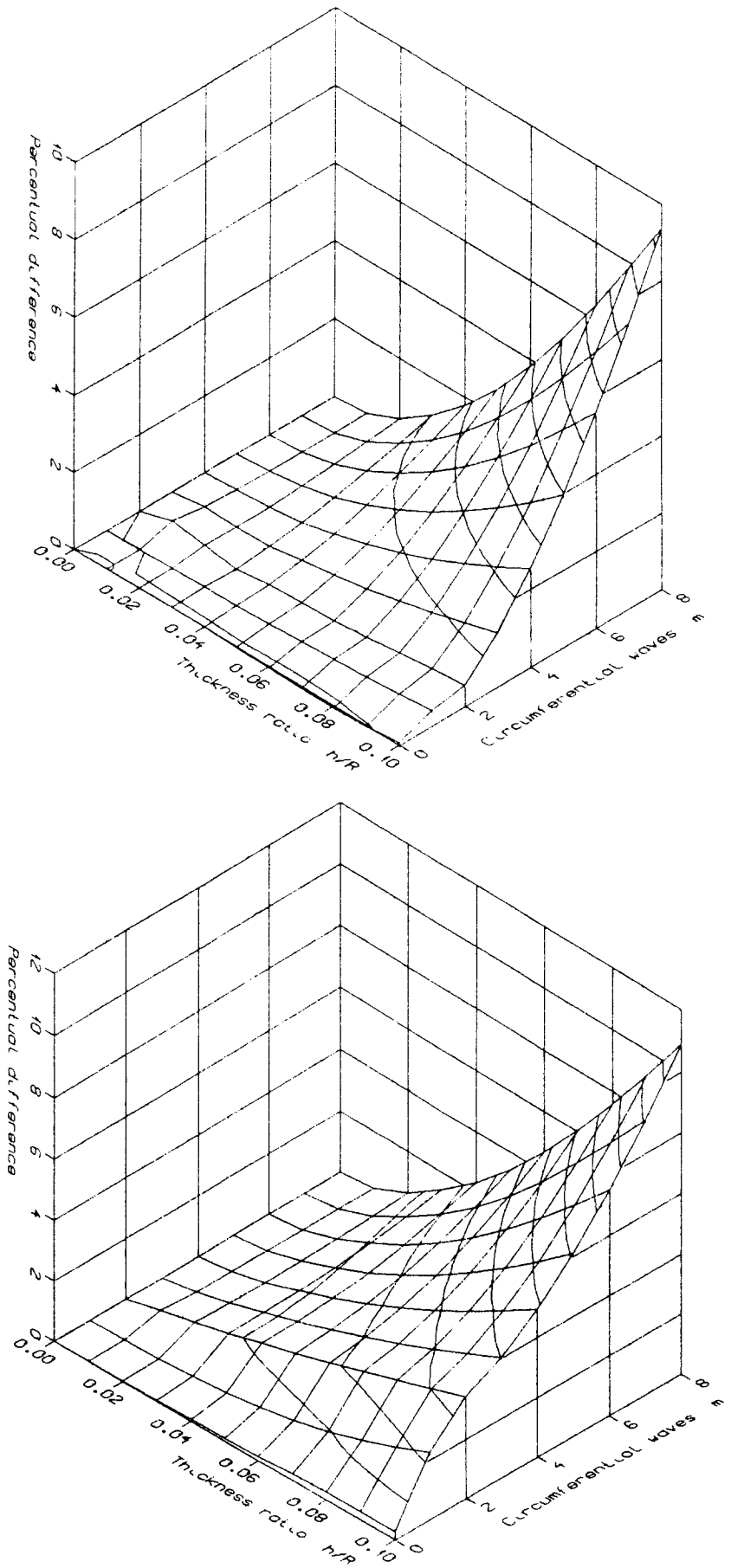

Fig. 5. Percentage errors ε obtained from the equations of Morley–Koiter: (a) symmetric mode; (b) skew-symmetric mode. Percentage isolines with 1% spacing are shown.

Table 5. The coefficients in the power expansion of the error, eqn (1). The upper three rows are the symmetric mode and the lower rows are the skew-symmetric mode: (a) refined equations; (b) Flügge; and (c) Morley–Koiter

Equation	A	B	C	D	E
(a)	−0.046	−0.052	0.055	0.75	0.73
(b)	−0.053	−0.086	0.061	1.1	0.50
(c)	−0.053	−0.070	0.058	1.2	0.43
(a)	0.26	0.68	−0.026	0.92	0.26
(b)	0.25	0.64	−0.018	1.3	0.007
(c)	0.25	0.52	−0.017	1.3	0.061

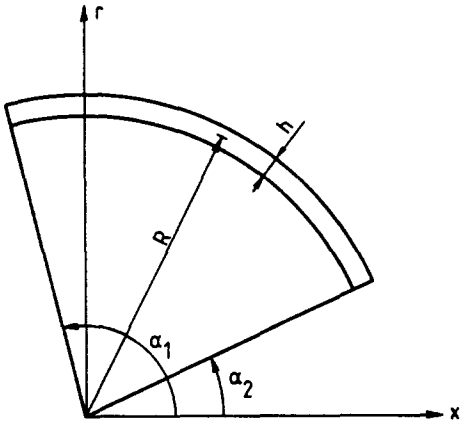

Fig. 6. Spherical shell.

Only a slight difference exists between the coefficients of the three sets of shell equations. The coefficient of $(h/R)^2$, i.e. A, is small in all cases. The coefficient of $(h/R)^2$, i.e. B, is not estimated *a priori* in the classical approach, but in the refined equations it is estimated to be of the order of one, which the value is well below. This is also observed in the classical equations. The coefficient of h/L, i.e. C, ought to be zero due to the *a priori* estimate. This is hardly so. The explanation is that the prescribed boundary conditions do not ensure that the edge tractions are zero. However, the magnitude of the coefficient is small. The coefficient of $(h/L)^2$, i.e. D, is predicted *a priori* to be of the order one and this is observed in Table 5. The coefficient of h^2/RL, i.e. E, is not explicitly predicted *a priori*.

Note, that in the case of axial symmetry ($m = 0$), the variation of h/R results in a very small frequency variation. This is due to the fact that the natural frequency here is independent of the bending stiffness of the shell. For all the cases considered the lowest overall natural frequency corresponds to motion involving two circumferential waves and the symmetric mode.

6. A FEW NUMERICAL RESULTS FOR A SPHERICAL ZONE

Now, we shall use a shell in the shape of a spherical zone described by two angles α_1 and α_2 (Fig. 6). This problem has recently been investigated by Niordson[15], who used Love's first approximation for the strain energy, i.e. in that sense, a classical approach. We compare the shell and FEM solutions for various values of h/R and fixed values of m, α_1 and α_2.

Here, the frequency is given in the form

$$\omega = c_N \frac{h}{R^2} \sqrt{\left(\frac{E}{2(1+\nu)\rho}\right)}. \tag{27}$$

The dimensionless coefficient c_N is shown in Table 6. The boundaries are free.

Table 6. The frequency coefficient c_N obtained from Ref. [15] (first row) and FEM (second row) for the first frequency, $\alpha_2 = 0°$. The percentage difference according to eqn (24) is also shown

		h/R					
α_1	m	0.01	%	0.02	%	0.04	%
60°	2	3.184		3.145		3.085	
		3.180	0.1	3.133	0.4	3.062	0.7
	3	8.132		7.929		7.647	
		8.114	0.2	7.884	0.6	7.556	1.2
	4	14.901		14.350		13.659	
		14.857	0.3	14.244	0.7	13.446	1.6
90°	2	2.081		2.052		2.007	
		2.071	0.5	2.042	0.5	1.990	0.8
	3	5.724		5.577		5.356	
		5.701	0.4	5.541	0.6	5.292	1.2
	4	10.817		10.411		9.834	
		10.773	0.4	10.330	0.8	9.689	1.5
120°	2	2.485		2.429		2.341	
		2.477	0.3	2.416	0.5	2.313	1.2
	3	7.300		7.025		6.608	
		7.277	0.3	6.979	0.7	6.511	1.5
	4	13.974		13.239		12.191	
		13.923	0.4	13.138	0.8	11.987	1.5
150°	2	7.429		6.989		6.315	
		7.397	0.4	6.932	0.8	6.215	1.6
	3	21.372		19.479		16.811	
		21.267	0.5	19.307	0.8	16.541	1.6
	4	38.917		34.489		28.538	
		38.745	0.4	34.177	0.9	28.095	1.5

Table 7. The coefficients of expansion (1)

α_1	A	B	C	D	E
60°	0.03	−2.7	0.23	−2.0	6.1
90°	0.36	−9.1	0.04	−1.8	8.7
120°	0.18	−2.1	0.12	−2.3	6.1
150°	0.38	−1.4	0.10	−2.4	1.8

We see that the error increases with increasing thickness to radius ratio. This behaviour is expected because of the *a priori* error estimate. By means of a least squares procedure as used in Section 5 we calculate the coefficients in the error expansion (1) from Table 6. This is shown in Table 7.

From this table we observe that A and D are of the order of one, as predicted by the *a priori* estimate.

7. CONCLUSION

From numerical investigations covering a wide range of geometrical and loading parameters, we conclude that the *a priori* estimate of the intrinsic errors agrees well with the true error regardless of the named equations employed, i.e. classical or refined.

Furthermore, we conclude that the use of a higher degree of accuracy in the derivation of the constitutive equations in the case of a circular cylindrical shell leads to better results compared with the classical equations. However, the classical equations yield remarkably good results, in spite of their much simpler form.

In the case of a spherical zone, we conclude that the agreement between the *a priori* error estimate and the true error is present.

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APPENDIX

In the case of a circular cylinder, the coupled constitutive equations are

$$N^{11} = H\left(u_{,x} + \nu v_{,\phi} + \frac{\nu}{R}w\right) + \frac{D}{2(1-\nu)R^3}(R^2\{(3\nu^2 + 2\nu - 2)w_{,xx} + (\nu^2 + 2\nu)w_{,\phi\phi}\} - (\nu^2 + 2\nu)(2Rv_{,\phi} - w))$$

$$N^{12} = \frac{1}{2}(1-\nu)H(u_{,\phi} + v_{,x}) + \frac{1}{2R^2}(1-\nu)D(Rw_{,x\phi} - v_{,x}) \tag{A1}$$

$$N^{22} = H\left(\nu u_{,x} + v_{,\phi} + \frac{1}{R}w\right) + \frac{D}{2(1-\nu)R^3}(R^2\{3\nu w_{,xx} + (4-\nu)w_{,\phi\phi}\} + (\nu - 4)(2Rv_{,\phi} + w))$$

$$M^{11} = D(w_{,xx} + \nu w_{,\phi\phi}) + \frac{D}{2(1-\nu)R^2}(R\{(4\nu^2 - \nu)v_{,\phi} + (3\nu^2 + 2\nu - 2)u_{,x}\} + (2\nu^2 + \nu)w)$$

$$M^{12} = (1-\nu)Dw_{,x\phi} + \frac{1}{4R}(1-\nu)D(u_{,\phi} - 3v_{,x}) \tag{A2}$$

$$M^{22} = D(\nu w_{,xx} + w_{,\phi\phi}) + \frac{D}{2(1-\nu)R^2}(R\{(\nu^2 + 2\nu)u_{,x} + 2\nu v_{,\phi}\} + (2+\nu)w)$$

where

$$H = \frac{Eh}{1-\nu^2}; \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{A3}$$

Substitution of the d'Alembert forces, eqns (15), and the displacement assumption, eqns (17), into the equilibrium equations leads to

$$\left(\alpha_{mj}^2 - \frac{1}{2}(1-\nu)m^2 + \Omega^2\right)u_{mj} + \frac{1}{2}\left(1 + \nu - k\frac{3\nu^2 + 2\nu + 1}{1-\nu}\right)\alpha_{mj}mv_{mj} + \left(\nu - \frac{k}{2(1-\nu)}\{\nu^2 + 2\nu - (3\nu^2 + 2\nu - 2)\alpha_{mj}^2 + (2\nu^2 + 1)m^2\} - k\Omega^2\right)\alpha_{mj}w_{mj} = 0$$

$$\frac{1}{2}\left(1 + \nu - k\frac{3\nu^2 + 2\nu + 1}{1-\nu}\right)m\alpha_{mj}u_{mj} - \left(\frac{1}{2}(1-\nu)\alpha_{mj}^2 - m^2 + k\left\{(1-\nu)\alpha_{mj}^2\right.\right. \tag{A4}$$

$$+\frac{2(\nu+2)}{1-\nu}m^2\Big\}+(1+6k)\Omega^2\Big)v_{mj}+\Big(1-\frac{k}{2(1-\nu)}\{\nu+8-(\nu^2+5\nu-3)\alpha_{mj}^2+3\nu m^2\}-3k\Omega^2\Big)mw_{mj}=0$$

$$\Big(\nu-\frac{k}{2(1-\nu)}\{\nu^2+2\nu-(3\nu^2+2\nu-2)\alpha_{mj}^2+(2\nu^2+1)m^2\}-k\Omega^2\Big)\alpha_{mj}u_{mj}+\Big(1-\frac{k}{2(1-\nu)}\{\nu+8$$

$$-(\nu^2+5\nu-3)\alpha_{mj}^2+3\nu m^2\}+3k\Omega^2\Big)mv_{mj}+\Big(1+k(\alpha_{mj}^2-m^2)^2-\frac{k}{1-\nu}\{3-(2\nu^2+\nu)\alpha_{mj}^2+(\nu+2)m^2\}$$

$$-(1+k(m^2-\alpha_{mj}^2))\Omega^2\Big)w_{mj}=0$$

with

$$\Omega^2=\frac{R^2(1-\nu^2)\rho}{E}\omega^2;\qquad k=\frac{h^2}{12R^2}. \tag{A5}$$